Empirical-Mechanistic Method Based Stochastic Modeling of Fatigue Damage to Predict Flexible Pavement Cracking for Transportation Infrastructure Management

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Abstract: In the purely theoretical approach of pavement design, percentage fatigue cracking is related to damage in a probabilistic manner according to the Miner’s law. Two methods that are currently widely in use are based on assumptions of damage distribution. One method assumes fatigue damage being normally distributed, while the other one assumes fatigue damage being lognormally distributed. Since mechanistic-empirical pavement design and pavement management require precise forecasting of pavement fatigue cracking, much effort should be taken to characterize and predict fatigue cracking in terms of damage distribution. In this paper, we formulate the probability density distribution of fatigue damage of flexible pavements according to the underlying structure of fatigue cracking equations so that pavement fatigue-cracking damage can be interpreted in a more meaningful way. Numerical computation is conducted for a case study. It is found that damage is neither normally nor lognormally distributed. It is therefore recommended that methodology and damage distribution model established in this paper be used in practice to predict damage distribution and percentage cracking so that a better estimation of fatigue cracking can be made.


CE Database keywords: Flexible pavements; Cracking; Damage; Transportation management; Stochastic processes; Density functions.

Introduction

The empirical-mechanistic (E-M) based method of pavement design is based on the mechanics of materials, which relates input such as a wheel loads to output such as pavement response. The response is then used to predict pavement distress (including cracking) and performance based on laboratory experiments and field testing. Field testing and adjustment is necessary because mechanistic theory alone has proven insufficient to predict pavement distress and performance (Huang 1993). In the E-M based methods of pavement design, a number of failure criteria related to specific distress and pavement performance must be established based on theory and field observations. This is in contrast to the American Association of State Highway and Transportation Officials (AASHTO) pavement design method, where most of the design criteria are empirically driven. Since the E-M based methods have a significant advantage over the purely empirical design methods, the trend of pavement design and management is to develop and adopt the E-M based design methods. It is actually

planned in the AASHTO Pavement Design Guide 2002 that pavement design methods should be E-M based procedures (Ali and Tayabji 1998).

As one of the important factors of E-M based pavement design, fatigue cracking has been reported as the most prevalent form of structure distress of flexible (i.e., asphalt concrete) pavements in the United States (Finn 1973). Generally, flexible pavement cracking can be classified into three categories: (1) traffic loading cracking; (2) low temperature cracking (Christison et al. 1972); and (3) thermal fatigue cracking. Among the many factors causing flexible pavement cracking, traffic loading, subgrade characteristics, and the environmental factors are the primary elements. Many flexible pavement design methods consider traffic load induced fatigue cracking as a major design criterion (Huang 1993).

In view of the fact that predicted pavement distress varies a great deal at the end of the designed servicing period, it is more reasonable to introduce probabilistic approaches to pavement design and management, since there is significant variability in predicting traffic loading, environmental conditions, and construction quality. An example of stochasticity is the lateral wander of traffic. Since wheel paths of different vehicles are not identical, lateral distribution of wheel paths should be considered in formulating the design traffic. Lemer and Moavenzadeh (1971), and Darter and Hudson (1973) were among the first to introduce the reliability concept to pavement design and management. Reliability concepts were also incorporated in the Texas flexible pavement design systems (Irick et al. 1987; Uzan et al. 1990) and in the AASHTO Design Guide (AASHTO 1993). Moavenzadeh and his associates (Moavenzadeh et al. 1974) also developed a computer program based on the probabilistic analysis of three-layer viscoelastic pavement systems. This computer program, which incorporated the concept of serviceability and reliability, was later
on modified by the Federal Highway Administration (Kenis 1977) and renamed VESYS. Several versions of the VESYS program have been developed (Lai 1977; Jordahl and Rauhut 1983).

Flexible pavement fatigue cracking is usually controlled by the maximum tensile stress at the bottom of the asphalt layer. A number of predictive models of fatigue cracking have been developed over the past three decades to characterize traffic load induced fatigue cracking. In general, predictive models relate the number of load repetitions to a certain response of pavement structures. These predictive models play crucial roles in the E-M based design method.

To predict fatigue cracking of flexible pavement, damage needs to be cumulated according to certain rules. The most popular rule of these rules is the Miner’s law. Cumulated damage is interpreted as degree of fatigue deterioration of flexible pavement due to traffic loading. Because significant variability exists in factors such as traffic loading, pavement materials, and construction quality, cumulated damage varies stochastically and spatially (i.e., damage at a specific location of pavement is fixed but varies in different locations). Damage distribution estimation is needed to relate damage to percentage cracking. In existing pavement design methods and pavement management practice (Hass et al. 1994; Hudson et al. 1997), the probability distribution of cumulated damage is often assumed to be either a normal distribution or a lognormal distribution, the purpose of which is to simplify the analysis. Nothing, however, has been found in the literature to examine whether these assumptions are reasonable.

Other approaches for predicting fatigue cracking involve establishing an empirical regression equation for cracking directly. The fatigue cracking prediction model developed by Jackson et al. (1996) for the South Dakota Department of Transportation takes the form of Fatigue Cracking Index \( = 100 - 0.11726 \cdot AGE^{2.2} \). The Fatigue Cracking Index ranges from 0 to 100, depending on the current age of the pavement, and is determined by expert opinion and regression analysis. Clearly, only pavement age is included in this regression-based predictive model, which makes it impossible to analyze the effects of traffic, climate, and pavement structures.

Aliand and Tayabji (1998) proposed an alternative method based on field data to predict fatigue cracking in terms of damage. They correlated damage ratio with percentage fatigue cracking using growth curves. It was found that the percentage cracking increases only slightly, usually far less than 20%, before the damage index reaches 1, then goes up very quickly when the damage index approaches 10 or more, and tends to be stationary at a level of 78%. Aliand and Tayabji (1998) then suggested that a curve used to regress the percentage cracking in terms of the damage index should allow for stabilization at 80% fatigue cracking; meanwhile, percentage fatigue cracking should be at a low severity level until the damage index approaches 100%. An obstacle imposed by this method is the requirement for field observation of percentage cracking. In most cases, field data are not readily available if one wants to predict fatigue cracks rather than to evaluate them.

Since these methods are not E-M based methods, they can only be applicable to a specific data set or area and cannot be directly used by others. Realizing the importance of fatigue cracking in pavement design and pavement management, it is necessary and worthwhile to develop a theoretically sound method of estimating damage distribution and characterizing fatigue-cracking progression in flexible pavements. Furthermore, a realistic damage distribution model can be a very useful tool in determining precisely the state transition probability in a Markov-chain based transportation infrastructure maintenance system, which can further contribute to the optimization of funding allocation. In view of these benefits, in this paper we formulate pavement damage distribution and percentage fatigue cracking based on existing predictive models of flexible pavement fatigue cracking. We also examine and compare assumption-based models (i.e., normal and lognormal distributions) with our theory-based model (i.e., the distribution derived in this paper) through a numerical example.

### Predictive Models of Fatigue Cracks

Before addressing the probability distribution of cumulated damage, it is necessary to review predictive fatigue-cracking models, because these models are closely related to pavement damage and its underlying statistical structure. Such a state-of-the-art review is also indispensable, because knowing how these predictive models are obtained is important for engineers to select appropriate models for their specific concern, such as climate and material considerations.

It is generally recognized that the allowable number of traffic load repetitions is closely related to tensile strain at the bottom of the asphalt layer. A universal form of the fatigue law used to predict fatigue-cracking life of flexible pavements (Finn 1973; Finn et al. 1973, 1977) is:

\[
N = k_1 e^{-k_2 E^{-k_3}}
\]

where \( e \) = maximum tensile strain at bottom of asphalt layer; \( E \) = resilient modulus (i.e., stiffness) of the asphalt layer; \( k_1,i (i = 1,2,3) \) are parameters of fatigue law; and \( N \) = total number of load repetitions to failure.

The coefficients \( k_2 \) and \( k_3 \) are calibrated through laboratory beam-type fatigue testing. Laboratory cyclic loading is generally applied continuously with very small, equal rest periods (Hoyt et al. 1987; Tseng and Lytton 1990). A significant difference exists between laboratory fatigue testing and field observation. One of the reasons for this is that in the field a random rest period exists between the successive loads, which allows the asphalt material to heal. Failure in laboratory tests is relatively sudden, soon after cracks initiate, whereas some maximum acceptable amount of surface cracks may occur in the field long after initial cracking at the bottom of the asphalt layer (Rauhut et al. 1975). Also, actual stress and strain responses of a pavement surface layer reticulated by cracks at the bottom may be considerably different from those estimated with elastic layer theory (Rauhut et al. 1975). The load applied in laboratory is not identical to that exerted in the field. Residual stresses caused by a moving wheel may remain in the asphalt surface layer after the passage of each load. These stresses relax with time, and after a sufficient time lapse, a smaller residual stress would remain. In the laboratory, the residual stress builds up in fatigue samples, and its magnitude is much different compared with that of the field (Hoyt et al. 1987; Sun and Deng 1998; Sun 1998; Tseng and Lytton 1990).

The laboratory fatigue life of an asphalt material is usually lower than that observed in the field. Laboratory fatigue life, therefore, must be adjusted by a shift factor to obtain the field fatigue life such that appropriate design criteria can be developed (Corte and Goux 1996; Das and Pandey 1999). This process may be considered as the modification of the laboratory result by the field-calibrated fatigue and is reflected by the coefficient \( k_1 \) of Eq. (1).

Major institutions that provide fatigue-cracking curves include the Asphalt Institute (AI) (1981), Shell International Petroleum...
The number of load repetitions required to progress from the onset of fatigue cracking is fewer for thin asphalt layers than for thicker layers, Craus et al. (1984) suggested that $k_1$ in the AI model be reduced to 0.0636 for hot mixture asphalt layers less than 4 in. (102 mm) in thickness. The AI model applies to a typical situation where the amount of asphalt cement in asphalt concrete is 5%. In cases where the condition is not satisfied, the following fatigue cracking curve can be adopted (AI 1982):

$$N = 0.07958 \times (10^{0.6}) e^{-3.291 E^{-0.854}}$$

in which $V_a$ and $V_b =$ percentage volume of air voids and asphalt, respectively AI 1982, (Aliand and Tayabji 1998).

Other simplified fatigue-cracking curves are also available in literature, such as the curves used in Illinois (Thompson 1987) and Minnesota (Timm et al. 1998), and the fatigue curve developed by Bergan and Pulles (1973) for cold climates. These fatigue-cracking curves are adopted by various highway agencies. Table 1 provides a brief summary of these fatigue-cracking models.

For those AI based fatigue-cracking models (e.g., the modified AI model), failure is defined as 45% fatigue cracking in the wheelpath area. For other fatigue-cracking models, their failure criteria may not be identical to the AI criterion. One of the possible failure criteria used in these models can be that failure is defined as the appearance of fatigue cracking. Consequently, as long as the damage ratio is equal to unity, fatigue cracking occurs according to the Miner’s law.

Figs. 1–3 plot these predictive fatigue-cracking models versus maximum tensile strain at the bottom of asphalt layer. It can be seen from Table 1 and Figs. 1–3 that the exponent $k_2$ of the fatigue equation varies slightly, but the coefficient $k_1$ varies over several orders of the magnitude. The exponents $k_2$ and $k_3$ are determined from fatigue tests on laboratory specimens. The coefficient $k_1$ is obtained by calibrating laboratory results according to field observations. Due to differences in materials, test methods, field conditions, and structural models, a large variety of fatigue equations are expected (Huang 1993). For instance, fatigue life in the Federal Highway Administration report of the Texas Flexible Pavement System (Uzan et al. 1990) is evaluated using the equation by Finn et al. (1977), which was modified to take into account the cracking propagation phase:

$$\log N = -3.13 + \frac{h}{380} - 3.291 \log e - 0.854 \log E$$

in which $h =$ thickness of the asphalt-concrete layer (mm).

In view of these predictive models of fatigue cracking, how to select an appropriate model for pavement design and management thus becomes crucial, especially because significant differences exist in these predictive models (NCHRP 1990). No agreement

### Table 1. Predictive Models of Flexible Pavement Fatigue Cracking

<table>
<thead>
<tr>
<th>Models</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>Sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>AI model</td>
<td>0.0796</td>
<td>3.291</td>
<td>0.854</td>
<td>Asphalt Institute (1981)</td>
</tr>
<tr>
<td>Shell model</td>
<td>0.0685</td>
<td>5.671</td>
<td>2.363</td>
<td>Shell Ltd. (Shell 1978; Shook et al. 1982)</td>
</tr>
<tr>
<td>Belgian Road Research Center</td>
<td>$4.92 \times 10^{-14}$</td>
<td>4.76</td>
<td>0</td>
<td>Verstraeten et al. (1984)</td>
</tr>
<tr>
<td>UC-Berkeley</td>
<td>0.0636</td>
<td>3.291</td>
<td>0.854</td>
<td>Craus et al. (1984)</td>
</tr>
<tr>
<td>Modified AI model</td>
<td>1984</td>
<td>3.0</td>
<td>2.66</td>
<td>Department of Defense (1988)</td>
</tr>
<tr>
<td>Transport and Road Research Laboratory</td>
<td>$1.66 \times 10^{-10}$</td>
<td>4.32</td>
<td>0</td>
<td>Powell et al. (1984)</td>
</tr>
<tr>
<td>Illinois model</td>
<td>$5 \times 10^{-6}$</td>
<td>5.0</td>
<td>2.66</td>
<td>Thompson (1987)</td>
</tr>
<tr>
<td>U.S. Army model</td>
<td>478.63</td>
<td>3.21</td>
<td>0</td>
<td>Timm et al. (1998)</td>
</tr>
<tr>
<td>Minnesota model</td>
<td>$2.83 \times 10^{-6}$</td>
<td>3.21</td>
<td>0</td>
<td>Das and Pandey (1999)</td>
</tr>
<tr>
<td>Indian model</td>
<td>0.1001</td>
<td>3.565</td>
<td>1.4747</td>
<td></td>
</tr>
</tbody>
</table>

(Shell 1978; Shook et al. 1982), the University of California at Berkeley (Finn 1973; Finn et al. 1973, 1977; Craus et al. 1984), the U.S. Army (Department of Defense 1988), and the University of Nottingham (Brunton et al. 1987).
has been reached among researchers to select the best predictive model. A feasible manner of selecting an appropriate predictive model for a transportation agency is to calibrate the model using field data that come exactly from regions where this predictive model is going to be used. In this way the calibrated predictive models become more suitable for a specific climate, environment, and traffic condition. For instance, N might be sensitive to each pavement site; therefore, it should be calibrated depending on the specific site (Uzan et al. 1991; Killingsworth and Zollinger 1995).

No matter which kind of predictive model of fatigue cracking is adopted, in the literature no experiment-based empirical evidence or procedure is currently available to convert damage in terms of the number of load repetitions obtained from these predictive models to percentage fatigue cracking, which is very important in providing a meaningful interpretation of damage in practice. The gap between damage and percentage fatigue cracking is usually filled with a very strong hypothesis artificially enforced. With the application of the Miner’s law, for example, fatigue-cracking damage can be accumulated over the summation of the load groups as well as over the time period. The method described by Uzan et al. (1991) predicts a single value for fatigue life in terms of N.

Modeling Pavement Fatigue Damage

Distribution of Total Number of Allowed Load Repetitions

Deterministic e and E

Because minor material or test variations in the exponent coefficient \( k_2 \) can cause major variations in the fatigue life of asphalt concrete pavements, due to the exponential nature of the relationship (this is actually reflected by the scatter fashion of the total number of load repetitions with respect to maximum strain during the laboratory fatigue tests), experiment data are converted on logarithm scales in pavement fatigue-cracking testing. The following equation is often used to represent the total number of load repetitions as a function of two primary variables, maximum strain \( \varepsilon \) and stiffness \( E \):

\[
\ln N = -k_2 \ln \varepsilon - k_3 \ln E + \ln k_1 + \varepsilon
\]

in which \( \varepsilon \) represents the error term of a regression model.

A predictive fatigue-cracking equation is obtained based on linear regression of experimental data. The parameters in model Eq. (1) are estimated using least-square estimation in Eq. (5). A basic assumption associated with linear regression models like Eq. (5) is that the error term is an independent random variable with normal distribution. This implies that if predictive fatigue-cracking models proposed in the past decades are correct according to standard statistical procedures, the error term \( \varepsilon \) should obey a normal distribution \( N(0, \sigma^2) \) where \( \sigma^2 \) (total variance), which interprets all the variations that are not explained by \( \varepsilon \) and \( E \).

With this argument in mind, it is straightforward to see the distribution of the left-hand side of Eq. (5), provided that \( \varepsilon \) and \( E \) are known and fixed:

\[
\ln N \sim N(\mu_\varepsilon, \sigma^2_\varepsilon)
\]

Here, \( \mu_\varepsilon = -k_2 \ln \varepsilon - k_3 \ln E + \ln k_1 \). One may notice that we use “N” to represent the normal distribution, and italic “\( N \)” to represent the total number of load repetition. Since Eq. (6) indicates that \( \ln N \) satisfies a normal distribution, it virtually says that the random variable \( N \) satisfies a lognormal distribution, i.e.

\[
N \sim \ln N(\mu_\varepsilon, \sigma^2_\varepsilon)
\]

It should be noted that the mean and variance of \( N \) are not given by \( \mu_\varepsilon \) and \( \sigma^2_\varepsilon \), but by

\[
\text{var } N = \exp[2(\mu_\varepsilon + \sigma^2_\varepsilon)] - \exp[2\mu_\varepsilon + \sigma^2_\varepsilon]
\]

Stochastic \( \varepsilon \) and E

The distribution Eq. (6) holds if maximum tensile strain \( \varepsilon \) and resilient modulus \( E \) are deterministic (i.e., fixed). This is true when pavement fatigue Eq. (1) was initially developed in pavement fatigue cracking testing, because these tests are conducted under the control of experimental design. In other words, level of strain and modulus are experimentally arranged rather than randomly observed. In reality, however, because of construction quality and other factors, both \( \varepsilon \) and \( E \) might be random variables with respect to the spatial location of pavement structures. To deal with this case, we rewrite Eq. (1) as

\[
N = k_1 A_k
\]

where \( A_k = A_2 A_3 \); \( A_3 = A_1 A_0 \); \( A_2 = \varepsilon^{-k_2} \); \( A_1 = E^{-k_3} \); and \( A_0 = \exp(\varepsilon) \). We first introduce the following theorems before we further evaluate the probability density function (pdf) of \( N \).

Theorem 1 (Casella and Berger 1990): Let \( X \) have pdf \( f_X(x) \) and let \( Y = g(X) \), where \( g \) is a monotone function. Let \( \Xi \) and \( \Psi \) be defined by \( \Xi = \{x; f_X(x) > 0\} \) and \( \Psi = \{y; y = g(x)\} \) for some \( x \in \Xi \). Suppose that \( f_Y(y) \) is continuous on \( \Xi \) and that \( g^{-1}(y) \) has a continuous derivative on \( \Psi \). Then the pdf of \( Y \) is given by

\[
f_Y(y) = \left\{ \begin{array}{ll}
\frac{d}{dy} g^{-1}(y) & y \in \Psi \\
0 & \text{otherwise}
\end{array} \right.
\]

Theorem 2: Given two independent random variables \( X \) and \( Y \), and their pdfs \( f_X(x) \) and \( f_Y(y) \), the pdf of the product \( W = XY \) can be given by

\[
f_W(w) = \int_0^w \frac{1}{y} f_X(w/y) f_Y(y) dy = \int_0^w \frac{1}{x} f_X(x) f_Y(1/x) dx
\]
be expressed in a closed form. Both cases, however, are studied in this section.

The transformation transforms the joint pdf \( f_{X,Y}(x,y) = f_X(x)f_Y(y) \) into the joint pdf \( f_{W,Y}(w,y) \):

\[
f_{W,Y}(w,y) = f_X(x)\frac{\partial x}{\partial w}f_Y(y)\frac{\partial y}{\partial y}
\]

where \( J = \text{Jacobian of the inverse transformation, Eq. (13):} \)

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial w} & \frac{\partial x}{\partial y} \\
\frac{\partial y}{\partial w} & \frac{\partial y}{\partial y}
\end{vmatrix} = \begin{vmatrix}
1 & -w/y^2 \\
0 & 1
\end{vmatrix} = 1/y
\]

Integrating out the variable \( y \) in Eq. (15) yields the marginal pdf, i.e., the pdf of the product \( w = xy \), namely, Eq. (12).

Let \( f_A(e) \), \( f_E(E) \), \( f_{A_1}(a_1), \ldots f_{A_4}(a_4) \), and \( f_{N}(n) \) be the pdfs of random variables \( e, E, A_1, \ldots A_4 \) and \( N \). Clearly, since \( e \sim N(0,\sigma^2_e) \), we know that \( A_0 \) obeys the lognormal distribution with ln \( A_0 \sim N(0,\sigma^2) \). According to Theorem 1, the pdfs of \( A_1 \) and \( A_2 \) can then be given in terms of the pdfs of \( e \) and \( E \), respectively:

\[
f_{A_1}(a_1) = f_E(a_1^{-1/k_1}) \frac{1}{k_1} a_1^{-1/(1+k_1)}
\]

\[
f_{A_2}(a_2) = f_E(a_2^{-1/k_2}) \frac{1}{k_2} a_2^{-1/(1+k_2)}
\]

Similarly, according to Theorem 2, the pdfs of \( A_3 \) and \( A_4 \) can be given in terms of the pdfs of \( A_1 \) and \( A_0 \), and \( A_2 \) and \( A_2 \), respectively:

\[
f_{A_3}(a_3) = \int_0^\infty f_{A_1}(a_1) f_{A_0}(a_0) \frac{a_3}{a_0} da_1 = \int_0^\infty f_E(a_1^{-1/k_1}) \frac{1}{k_1} a_1^{-1/(1+k_1)} \frac{a_3}{a_0} da_1
\]

\[
f_{A_4}(a_4) = \int_0^\infty f_{A_2}(a_2) f_{A_0}(a_0) \frac{a_4}{a_0} da_2 = \int_0^\infty f_E(a_2^{-1/k_2}) \frac{1}{k_2} a_2^{-1/(1+k_2)} \frac{a_4}{a_0} da_2
\]

Using Theorem 1 again, we can express the pdf of \( N \) based on the results of Eqs. (16)–(19):

\[
f_{N}(n) = \frac{1}{k_1} f_{A_1} \left( \frac{n}{k_1} \right)
\]

So far we have obtained the pdf of \( N \) in both the deterministic and stochastic cases. In the deterministic case, the total number of allowed loading repetitions \( N \) is lognormally distributed, while in the stochastic case, the pdf of \( N \) does not belong to any known distribution, though its form can still be given analytically by Eq. (20). The computational complexity involved in calculating the pdf Eq. (20) is higher than that given by Eq. (6). Even if \( e \) and \( E \) are stochastically valued, as long as the variation associated with \( e \) and \( E \) is not significant, Eq. (6) can still be used as an approximation. In the following sections only the pdf given by Eq. (6) is adopted so that the result can be derived in a closed form. It should be pointed out that there is no difficulty in derivation when adopting the pdf given by Eq. (20), except that the result cannot be expressed in a closed form. Both cases, however, are studied numerically in a following section.

**Distribution of Applied Traffic Loads**

Traffic load application can be properly modeled as a random variable with a Poisson distribution in many situations. The same assumption is used in VESYS for modeling traffic loading (FHWA 1983; Huang 1993). For a Poisson distributed random variable, its probability mass function is given by

\[
P(X=x|\lambda) = e^{-\lambda}x! / x! \quad x=0,1,\ldots; \quad 0\leq\lambda<\infty
\]

where \( X \) represents the traffic variable, and \( \lambda = \text{ratio of a Poisson arrival process, which can be obtained from traffic prediction. The mean and variance of a Poisson random variable are identical. It is known that when the arrival (i.e., cumulated traffic) in a Poisson distribution is greater than 30, the probability of arrival being greater than a certain value can be well approximated by a normal distribution. In practice, the estimated traffic loading is usually much larger than 30; therefore, such an approximation would be rather precise. In other words, we can also model the traffic loading distribution as a normal distribution with the same mean and variance; that is}

\[
\mu_{\text{traffic}} = \text{EX} = \sigma^2_{\text{traffic}} = \text{VarX} = \lambda T
\]

in which \( T = \text{time up to which traffic loading is to be predicted. We may also directly assume that traffic loading is normally distributed with a mean and variance estimated directly from field observations.}

**Damage Distribution**

So far we have already characterized the distributions of traffic loading and the allowable total number of load repetition. The amount of damage is indicated by the so-called damage index, which is the ratio between the predicted and allowable number of load repetitions. To accumulate fatigue-cracking damage due to a variety of load applications and environmental effects, the Miner’s law of cumulative damage (Miner 1945) has been widely adopted in many pavement design procedures. The general form of the Miner’s law is

\[
D = \sum_{i=1}^{m} \frac{X_i}{N_i}
\]

where \( D = \text{cumulated damage in the surface layer; } X_i = \text{actual number of traffic repetitions applied to the pavement during period } i \); and \( N_i = \text{total number of traffic load repetitions that can be allowable by pavement during period } i \).

Two hypothetical damage distributions (FHWA 1983; Huang 1993), i.e., normal and lognormal distributions, are currently widely adopted in many flexible pavement design methods. Based on field observations, Aliand and Tayabji (1998) have seen that damage index \( D \) can be as large as 17–20 given that the percentage cracking is less than 100%. According to Eq. (23), it is clear that damage index \( D \) is neither normally nor lognormally distributed. To obtain damage distribution, we now only consider the damage index as the ratio of total traffic loading over allowable total number of load repetition for simplicity purposes, i.e., \( D = X/N \), which is also used in VESYS for estimating the first and second moments of damage. It should be noted that \( D = X/N \) does not mean that only one value of \( N \) exists for each specific pavement structure. Instead, \( N \) is probabilistically distributed over a range of values.

Define a new random variable \( G = N \). The inverse functions of \( G \) and \( D \) can be, respectively, given by...
According to algebra of random variables (Casella and Berger 1990), the joint pdf of a bivariate function can be given by

\[ f_{D,G}(d,g) = f_{X,N}[h_1(d,g), h_2(d,g)] | J | \]

(26)

in which \( f_{X,N}(\cdot) = \text{joint distribution of random variables } X \text{ and } N \); and \( J = \text{Jacobian given by} \)

\[ J = \begin{vmatrix} \frac{\partial X}{\partial d} & \frac{\partial X}{\partial g} \\ \frac{\partial N}{\partial d} & \frac{\partial N}{\partial g} \end{vmatrix} = g \]

(27)

Assume that \( X \) and \( N \) are independent random variables, which is usually true because actual traffic application is statistically independent of the allowable total number of load repetitions. The joint pdf of \( f_{X,N}(\cdot) \) can be expressed as the product of the individual pdf of \( X \) and \( N \)

\[ f_{D,G}(d,g) = \frac{1}{\sqrt{2\pi \sigma_{\text{traffic}}}} \exp \left\{ - \frac{(gd - \mu_{\text{traffic}})^2}{2\sigma_{\text{traffic}}^2} \right\} \]

\[ \cdot \frac{1}{\sqrt{2\pi \sigma_e}} \exp \left\{ - \frac{(\ln g - \mu_e)^2}{2\sigma_e^2} \right\} \]

(28)

The first factor on the right-hand side of Eq. (28) is the pdf of a normal distribution, and the second factor is the pdf of a lognormal distribution. Since the joint pdf of \( f_{D,G}(d,g) \) has been obtained, it is straightforward to get the pdf of damage by means of marginal distribution, i.e.

\[ f_D(d) = \frac{1}{2\pi \sigma_{\text{traffic}} \sigma_e} \int_0^\infty \exp \left\{ - \frac{(gd - \mu_{\text{traffic}})^2}{2\sigma_{\text{traffic}}^2} \right\} \]

\[ - \frac{(\ln g - \mu_e)^2}{2\sigma_e^2} \right\} dg \]

(29)

Eq. (29) gives the general expression of the pdf of damage distribution. It indicates that damage distribution is actually a function of the distributions of traffic loading and the allowable total number of load repetitions. If the analysis period is divided into a number of subintervals, the distribution of the cumulated damage can still be obtained based on a similar analysis to that preceding. To this end, define \( D_i = X_i / N_i \). The cumulated damage is thus given by \( \Sigma_m^{i=1} D_i \). Define transformation \( U_1 = D_1 + D_2 + \ldots + D_m \), \( U_2 = D_2 + \ldots + D_m \), \( U_3 = D_3 + \ldots + D_m \), \ldots, \( U_m = D_m \), and its inverse \( D_1 = U_1 - (U_2 + \ldots + U_m) \), \( D_2 = U_2 - (U_3 + \ldots + U_m) \), \ldots, \( D_m = U_m \). Similar to Eq. (26), the multivariate joint distribution \( f_V(u_1, u_2, \ldots, u_m) \) can be given by

\[ f_V(u_1, u_2, \ldots, u_m) = f_D(d_1, d_2, \ldots, d_m) | J | \]

(30)

where \( \mathbf{U} = (U_1, U_2, \ldots, U_m) \); \( \mathbf{D} = (D_1, D_2, \ldots, D_m) \); and the Jacobian is

\[ J = \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \]

(31)

Given that \( D_i (i = 1, 2, \ldots, m) \) are mutually independent, damage distribution can then be given by the marginal distribution of \( f_V(u_1, u_2, \ldots, u_m) \):

\[ f_D(d) = \int_0^\infty \int_0^\infty f_D(u_1 - u_2 - u_3 - \ldots - u_m) \]

\[ \times f_{D_2}(u_2) \ldots f_{D_m}(u_m) du_2 du_3 \ldots du_m \]

(32)

where the pdf \( f_D(\cdot) \) is given by Eq. (29). Integration of Eq. (32) is a general result that formulates the distribution of cumulated damage. Whether Eq. (29) or Eq. (32) should be used for modeling damage distribution depends upon the traffic spectrum. For instance, if there is only one component of the traffic loading spectrum, then Eq. (29) can be appropriate. Otherwise, if the traffic spectrum consists of many type of loadings, Eq. (32) ought to be used.

### Modeling Percentage Fatigue Cracking

In the previous section, we established the theory of damage distribution. Damage caused by traffic loading can be now readily calculated from the linear accumulation of the damage caused by individual load application. It is, however, preferred to express fatigue cracking in a more intuitive manner. In other words, it would be more meaningful if the damage index can be converted to the percentage of fatigue cracking. The purpose of assuming or modeling damage distribution is that it provides a mechanistic-empirical way of predicting pavement fatigue cracking. This is important in practice because both pavement management and pavement design need to be able to capture the quantitative characteristics of pavement fatigue cracking and its progress.

It has been widely accepted in pavement engineering practice that failure in terms of fatigue cracking occurs when the cumulative damage \( D \) reaches or exceeds 1 (i.e., Miner’s law). In other words, the percentage fatigue cracking can be given by

\[ \% C = 100 \cdot \text{Prob}(D \geq 1) \]

(33)

where \( \% C \) represents percent cracking.

We see from Eq. (33) that the percent of fatigue cracking in the pavement is computed as the probability that the damage index is surpasses a critical value 1. With one traditional assumption, suppose that damage is normally distributed, i.e., \( D \sim N(\mu, \sigma^2) \). The percentage cracking can be given by

\[ \% C = 100 \cdot \Phi \left( \frac{\mu - 1}{\sigma_D} \right) \]

(34)

where \( \mu_D \) and \( \sigma_D \) = mean and standard deviation of damage, respectively; and \( \Phi(z) \) = standard normal distribution with mean zero and variance 1. With another traditional assumption, suppose the damage is lognormally distributed, \( D \sim \ln N(\mu, \sigma_e^2) \). Then the logarithm of damage becomes normally distributed. The probability that damage is greater than 1 is equivalent to the probability that the logarithm of damage is greater than \( \ln 1 = 0 \). Given that the mean and variance of damage \( \mu_D \) and \( \sigma_D^2 \) are known, we can convert them using Eqs. (8) and (9) to obtain parameters of a lognormal distribution, i.e., \( \mu_e \) and \( \sigma_e^2 \). Eq. (35) gives the resulted characterizing parameters for a lognormal distribution:

\[ \sigma^2 = \ln \left( 1 + \frac{\sigma_D^2}{\mu_D^2} \right) \quad \text{and} \quad \mu = \ln \mu_D - \frac{1}{2} \ln \left( 1 + \frac{\sigma_D^2}{\mu_D^2} \right) \]

(35)

Percentage cracking can then be given by
\[
\% C = 100 \cdot \Phi \left( \frac{\mu}{\sigma} \right)
\]  \hspace{1cm} (36)

Now if we adopt the damage distribution obtained from this paper based on the bivariate transformation, the derived distribution is

\[
% C = 1 - \frac{1}{2\pi \sigma_{\text{traffic}} \sigma_e} \int_0^{\infty} \exp \left[ -\frac{(g_f - \mu_{\text{traffic}})^2}{2\sigma_{\text{traffic}}^2} \right] d\sigma_e
\]

\[
\left( \ln \frac{g_f - \mu_{\text{traffic}}}{2\sigma_e} \right)^2 d\sigma_e
\]

\[
= 1 - \frac{1}{\sqrt{2\pi} \sigma_{\text{traffic}}} \int_0^{\infty} \Phi \left( \frac{g_f - \mu_{\text{traffic}}}{\sigma_{\text{traffic}}} \right) d\sigma_e
\]

\[
- \Phi \left( \frac{-\mu_{\text{traffic}}}{\sigma_{\text{traffic}}} \right) \exp \left[ -\frac{(\ln g_f - \mu_e)^2}{2\sigma_e^2} \right] d\sigma_e
\]

In the derivation of integration Eq. (37), we exchange the order of the integration and use the transformation \( g_f = p \), \( gdf = dp \). Clearly, percentage cracking is a function of parameters \( \mu_{\text{traffic}}, \sigma_{\text{traffic}}, \mu_e, \) and \( \sigma_e \).

**Numerical Computation**

To illustrate what damage distribution looks like, numerical computation is conducted to serve this purpose. In the meanwhile, we want to examine whether traditional assumptions (i.e., normal distribution and lognormal distribution) are good enough for predicting percentage cracking. Without loss of generality, an asphalt concrete pavement is chosen to demonstrate the method developed in this paper. Here, we use Eq. (29) to compute the numerical value of damage.

First of all, two parameters \( \mu_D \) and \( \sigma_D^2 \) need to be estimated in the case of fatigue cracking damage assumed as a normal or lognormal distribution. According to Taylor’s expansion, mean and variance of damage can be obtained by means of the Cornell’s first-order, second-moment method (FHWA 1983; Uzan et al. 1991; Huang 1993; Killingsworth and Zollinger 1995). Under the assumption of independence of \( N \) and \( \tau \), we have

\[
\mu_D = ED = EX/EN = \mu_{\text{traffic}}/\exp(\mu_e + \sigma_e^2/2)
\]  \hspace{1cm} (38)

Here, \( EN \) in Eq. (38) has been replaced by Eq. (8). Also, the variance of damage can be given by

\[
\sigma_D^2 = \text{Var}(D) = \frac{\sigma_{\text{traffic}}^2}{(EN)^2} + \frac{\mu_{\text{traffic}}^2}{(EN)^4} \text{Var}(N)
\]  \hspace{1cm} (39)

where \( EN \) and \( \text{Var}(N) \) are given by Eqs. (8) and (9), respectively.

For a typical asphalt concrete pavement, let the resilient modulus of the asphalt layer \( E = 5.3 \times 10^5 \) psi and \( \epsilon = 3.45 \times 10^{-4} \). We choose the Shell model provided in Table 1 to calculate the distribution of the total number of load repetitions for this case study. Since \( \mu_e = -k_2 \ln \epsilon \) and \( k_2 \ln E + \ln k_1, \mu_e = 11.38263. \) Assume the standard deviation of total number of load repetitions \( \sigma_e = 0.08\mu_e = 0.910581. \) Also assume the predicted traffic \( \mu_{\text{traffic}} = 60,000 \) and \( \sigma_{\text{traffic}} = 245. \) Based on Eqs. (38) and (39), it is straightforward to obtain the mean and variance of damage, \( \mu_D = 0.451697 \) and \( \sigma_D = 0.513309 \) and \( \mu_e = 1.029333 \) and \( \sigma_e = 0.910585. \) After this, we can compute damage distribution in three cases: normal, lognormal, and derived.

Fig. 4 shows the pdfs of normal, lognormal, and derived distributions provided that the modulus and tensile strain be set up at fixed values, i.e., \( E = 5.3 \times 10^5 \) psi and \( \epsilon = 3.45 \times 10^{-4} \). We can see from Fig. 4 that the pdf of lognormally distributed damage tends to decay drastically when the damage index increases. This also applies to the pdf of normally distributed damage, though its decreasing tendency is not as strong as the lognormal distribution. The derived distribution shows a wider range distribution of damage where the probability of damage having large values cannot be neglected.

Fig. 5 shows the pdfs of derived distributions provided that the modulus and tensile strain be set up at fixed values, i.e., \( E = 5.3 \times 10^5 \) psi and \( \epsilon = 3.45 \times 10^{-4}, \epsilon \sim N(530,000, 100,000^2), \) and \( e \sim N(0.000345, 0.00008^2), E \sim N(530,000, 50,000^2), \) and \( e \sim N(0.000345, 0.000004^2). \) Interestingly, pdfs of damage do not vary very much in high magnitude ranges, say, \( D \geq 1, \) when variability associated with the modulus and tensile strain increases.

A comparison between the assumption based and the theory based pdf of damage illustrates that both the normal and the lognormal distribution assumptions may cause a large error in predicting variability associated with damage. To further illustrate
this, Fig. 6 plots the cumulative distribution density of damage. According to the Miner’s law, pavement cracks when the damage ratio exceeds unity. The percentage cracking is calculated as 14.1, 9.0, 33.8, 16.6, and 38.9%, respectively, corresponding to series 1–5.

Discussion

There have been several other methods proposed, such as nonlinear accumulation, to accumulate damage. Nevertheless, it is very difficult to actually calibrate parameters required by nonlinear models. Currently the Miner’s law is still widely accepted and adopted in practice (FHWA 1983; Huang 1993); therefore, we decided to adopt it in our study as well. It should also be noted that the traffic loading modeling in this paper is crude, because it has not considered different traffic distributions and components. Because that topic is complicated and needs more space to address, we will separately present it elsewhere in the future. To validate the proposed damage distribution model, the results obtained in this paper should be compared with actual field data, which will also be the work of further study.

Conclusions

In this paper we presented a theory of modeling pavement damage and predicting fatigue cracking using E-M based methodology and statistical theory. Probability density functions of flexible pavement damage are derived strictly according to the underlying structure of involved random variables. Explicit expressions of the pdf of damage and percentage cracking are presented. The pdf of damage shows that damage is neither normally distributed, as assumed in VESYS, nor lognormally distributed, as assumed in many other studies. It was found from numerical computation that damage tends to be distributed over a broad range of magnitudes. It is recommended that the damage distribution derived in this paper be used in practice so that a better estimation of fatigue cracking can be made, based on which transportation agency can optimize their pavement maintenance strategy to maximize the benefits to the traveling public.

Fig. 6. Cumulative density functions of damage [series 1: normal distribution; series 2: lognormal distribution; series 3: derived distribution as \( E = 5.3 \times 10^3 \) and \( \epsilon = 3.45 \times 10^{-4}; \) series 4: derived distribution, \( E \sim N(5.3 \times 10^3, 100,000^2) \) and \( \epsilon \sim N(3.45 \times 10^{-4}, 0.00008^2); \) series 5: derived distribution, \( E \sim N(5.3 \times 10^3, 50,000^2) \) and \( \epsilon \sim N(3.45 \times 10^{-4}, 0.00004^2)\)]

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References


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