The Bjorken $x$ variable in deep inelastic scattering – some additional motivation

Equation (7.6) shows that the cross section for deep inelastic scattering can be written as a function of two Lorentz-invariant quantities, $Q^2$ and $v$ (note that the Mott cross section is a function of $Q^2$ alone.) In some contexts (e.g. Fig. 7.1) it is useful to express things as a function of $W$, the “invariant mass” of the final-state hadrons, where $W$ can be written in terms of $Q^2$ and $v$ (see Eq. (7.1).) Eq. (7.7) introduces yet another Lorentz-invariant variable,

$$x = \frac{Q^2}{2P \cdot q} = \frac{Q^2}{2Mv}$$

$\quad (M = \text{proton mass}) \quad (7.7)$

The motivation for defining $x$, eventually discussed in Section 7.3, is the “parton model,” in which the virtual photon is assumed to interact with just one of several point-like constituents of the nucleon, as in Fig. 7.6. Some of the discussion of pages 86-87 is easier to follow if we make use of this motivation instead of the more formal treatment on page 86.

Assume that initially the interacting parton has a 4-momentum $xP$ which is a fraction $x$ of the 4-momentum $P$ of the nucleon, and that during the interaction this parton absorbs the entire 4-momentum $q$ of the virtual photon. Then the final 4-momentum of the parton is $P' = xP + q$, whose absolute square is equal to the parton mass squared. If the parton mass is small compared to the other energies and momenta in the system, then

$$m_{\text{parton}}^2 = |xP + q|^2 = x^2 P \cdot P + 2xP \cdot q - q^2 = x^2 M^2 + 2xP \cdot q - Q^2 = 0. \quad (1)$$

For large $Q^2 \quad (Q^2 \gg x^2 M^2)$, this gives Eq. (7.7).

Suppose the parton is point-like. Then the Fourier transform of its charge distribution is constant, i.e. independent of $Q^2$. But the structure functions $W_1$ and $W_2$ of Eq. (7.6) are relativistic analogues of the elastic form factor $F(q)$, which is the Fourier transform of the charge distribution (see Eq. (5.42).) Then, for a given $x$, the structure functions should be independent of $Q^2$. Since $x$ is dimensionless (see (7.7), or remember its interpretation as the fraction of the nucleon 4-momentum carried by the interacting parton), one can describe this feature as a scaling behavior: for sufficiently large values of $Q^2$, the structure functions depend only on the ratio of $Q^2$ to $v$, not on $Q^2$ or $v$ separately. This kind of scaling behavior is characteristic of interactions with point-like constituents, since the parton has no spatial dimensions to set an energy scale.

(For this reason, $x$ is known as the Bjorken scaling variable.) In order to test this scaling behavior, it is helpful to replace the structure functions $W_1$ and $W_2$ of Eq. (7.6), which have dimensions of $1/\text{energy}$, by dimensionless structure functions

$$F_1(x, Q^2) = \frac{M W_1(Q^2, v)}{Q^2, v} \quad (7.12)$$

Now assume that the nucleon is made up of $n$ point-like partons which are similar in their behavior. Then, on the average, each parton will possess one $n^{th}$ of the nucleon’s initial 4-momentum, and the average value of $x$ for deep inelastic scattering will be about $1/n$. (This agrees with the conclusions of Eq. (7.10) and (7.11).) If the number of partons is 3, then one expects that at sufficiently high $Q^2$ (where the approximations above are valid) the structure functions will show a maximum in the vicinity of $x = 1/3$, as seen in Fig. 7.2c.